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Qualitatively Robust Nonparametric Regression By Support Vector Machines

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Nonparametric Regression

$$Y = f_0(X) + \varepsilon$$

with

- Y : output variable (observable)
- X : input variable (observable)
- f_0 : regression function (unknown)
- ε : error term (not observable)

Goal: Estimation of the unknown regression function f_0

Qualitative Robustness

Qualitative Robustness – Parametric Statistics

Parametric Model

$$(P_\theta)_{\theta \in \Theta}\,, \qquad \Theta \subset \mathbb{R}^d$$

and

$$Z_i \;\sim\; P_{\theta_0} \quad \text{i.i.d.}\,, \qquad \theta_0 \in \Theta$$

Estimator

$$S_n \;:\; \mathcal{Z}^n \;\to\; \Theta\,, \qquad (z_1, \ldots, z_n) \;\mapsto\; S_n(z_1, \ldots, z_n)$$

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Robustness: different notions

$$\left.\begin{array}{l}\text{small model violations}\\ \text{small errors in the data}\end{array}\right\}\ \text{should not change the result too much}$$

Qualitative Robustness – Parametric Statistics

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Qualitative Robustness: Hampel (1968)

- Small errors in the data
  - Small errors in many of the data points (rounding etc.)
  - Large errors in a few data points (gross errors, outliers)

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- "should not change the result too much"

  P_θ : model distribution

  Q : real distribution

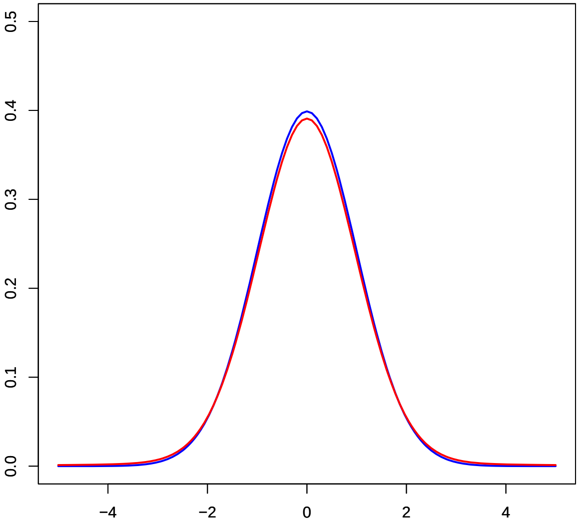
$$Q \text{ close to } P_\theta \qquad \longrightarrow \qquad S_n(Q^m) \text{ close to } S_n(P_\theta)$$

$S_n(P_\theta)$: assumed distribution of the estimator

$S_n(Q^m)$: real distribution of the estimator

(distribution of the estimator $=$ performance of the estimator)

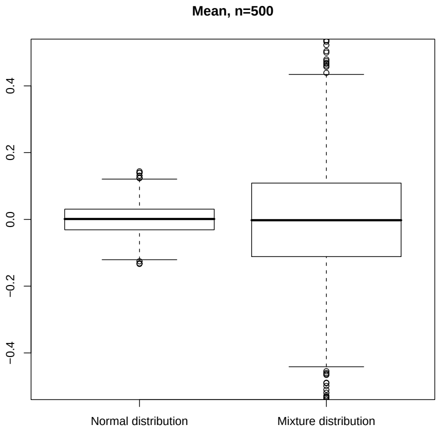
Qualitative Robustness – Example



Qualitative Robustness – Example

"mean" applied in 1000 runs

each run consists of a sample with 500 data points



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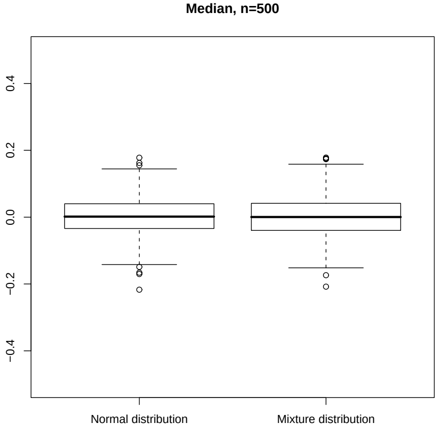

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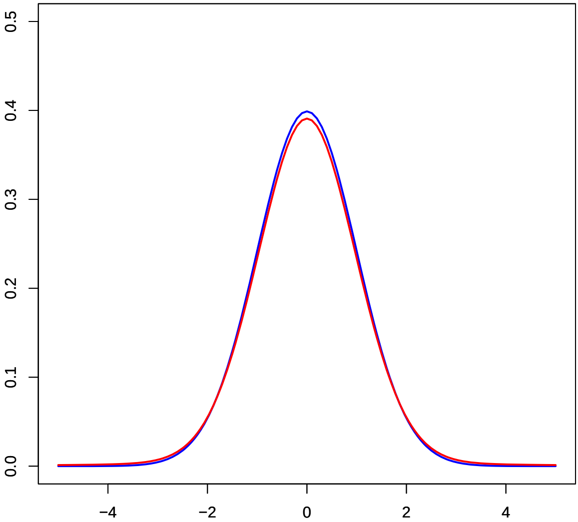

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The sequence of estimators $(S_n)_{n\in\mathbb{N}}$ is qualitatively robust if

$$\forall\, P\;\; \forall\, \epsilon > 0\;\;\; \exists\, \delta > 0 \;\text{ such that }\; \forall\, Q \;\text{ with }\; d_{\text{Pro}}(Q, P) < \delta$$

$$\sup_{n\in\mathbb{N}} d_{\text{Pro}}\big(S_n(Q^n), S_n(P^n)\big) < \varepsilon$$

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Prokhorov distance between probability measures Q and P:

$$d_{\text{Pro}}(Q,P) = \inf\big\{\delta > 0 \,\big|\, Q(A) \le P(A^\delta) + \delta \;\forall\, A\big\}$$

$$A^\delta = \big\{z \,|\, \inf_{z'\in A} d(z,z') < \delta\big\}$$

Qualitatively Robust Nonparametric Regression

Nonparametric Regression

$$Y_i \;=\; f_0(X_i) + \varepsilon_i\,, \qquad i \in \{1, \ldots, n\}$$

and

$$(X_i, Y_i) \;\sim\; P \qquad \text{i.i.d.}$$

Goal: Estimation of $f_0 : \mathcal{X} \to \mathcal{Y}$

Estimator:

$$S_n \;:\; (\mathcal{X} \times \mathcal{Y})^n \;\to\; H$$

H: a large set of functions $f : \mathcal{X} \to \mathcal{Y}$

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- Loss function

$$L \;:\; \mathcal{Y} \times \mathbb{R} \;\to\; [0, \infty)$$

$L(y, t)$: loss caused by estimation $t = \hat{f}_n(x)$ if y is true

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- Risk of an estimate $\hat{f}_n : \mathcal{X} \to \mathbb{R}$

$$\int L\big(y, \hat{f}_n(x)\big) \, P\big(d(x,y)\big)$$

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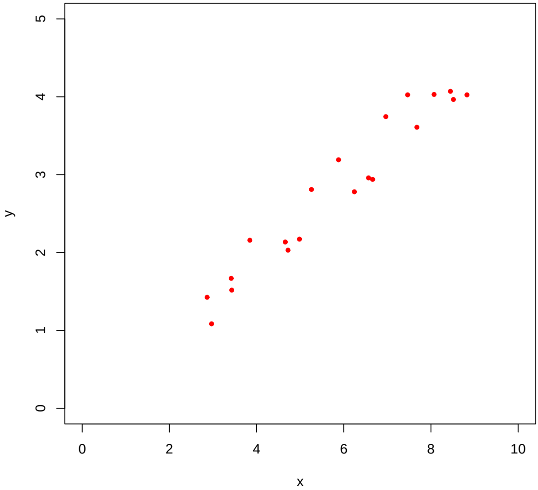
Overfitting



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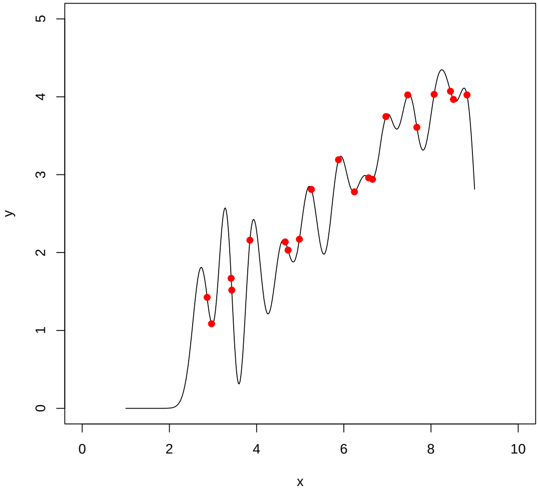

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Qualitative Robustness and SVMs

Support Vector Machines

$$\begin{aligned} S_n \ : \quad (\mathcal{X} \times \mathcal{Y})^n \quad &\rightarrow \quad H\,, \\ \big((x_1, y_1), \ldots, (x_n, y_n)\big) \quad &\mapsto \quad \arg\inf_{f \in H} \ \frac{1}{n} \sum_{i=1}^{n} L\big(y_i, f(x_i)\big) \ + \ \lambda \|f\|_H^2 \end{aligned}$$

with H a Hilbert space of functions $f : \mathcal{X} \rightarrow \mathbb{R}$

Qualitative Robustness:

A sequence of estimators $(S_n)_{n \in \mathbb{N}}$ is called qualitatively robust if

$$\forall\, P \ \ \forall\, \epsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } \ \forall\, Q \ \text{ with } \ d_{\text{Pro}}(Q, P) \ < \ \delta$$

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Qualitative Robustness and SVMs

Cuevas (1988):
If a sequence of estimators

$$S_n \ : \quad (\mathcal{X} \times \mathcal{Y})^n \ \rightarrow \ H \, , \qquad n \in \mathbb{N} \, ,$$

can be represented by a functional

$$S \ : \ \mathcal{M}_1(\mathcal{X} \times \mathcal{Y}) \ \rightarrow \ H \, , \qquad P \ \mapsto \ S(P) \, ,$$

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i.e.

$$S_n\big((x_1, y_1), \ldots, (x_n, y_n)\big) \ = \ S\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{(x_i, y_i)}\right) \quad \forall \, (x_i, y_i) \ \in \ \mathcal{X} \times \mathcal{Y}$$

for every $n \in \mathbb{N}$

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then

the sequence of estimators $(S_n)_{n \in \mathbb{N}}$ is qualitatively robust.

Qualitative Robustness and SVMs

In case of support vector machines

$$S_n\big((x_1,y_1),\ldots,(x_n,y_n)\big) \;=\; \arg\inf_{f\in H}\; \frac{1}{n}\sum_{i=1}^{n} L\big(y_i, f(x_i)\big) + \lambda\|f\|_H^2\,,$$

put

$$S\big(P\big) \;=\; \arg\inf_{f\in H} \int L\big(y, f(x)\big)\, P\big(d(x,y)\big) + \lambda\|f\|_H^2 \qquad \forall\, P\,.$$

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It remains to be proven:

$$P_n \xrightarrow[n \to \infty]{} P_0 \;\text{ weakly} \quad \Rightarrow \quad \Big\| S(P_n) - S(P_0) \Big\|_H \xrightarrow[n \to \infty]{} 0$$

Qualitative Robustness and SVMs

Assumptions:

- $\mathcal{X}$ a Polish space, $\mathcal{Y} \subset \mathbb{R}$ closed
- $L : \mathcal{Y} \times \mathbb{R} \to [0, \infty)$ continuous
  - $t \mapsto L(y, t)$ convex for every $t \in \mathbb{R}$
  - $t \mapsto L(y, t)$, $y \in \mathcal{Y}$, uniformly Lipschitz continuous
- the reproducing kernel

$$k \; : \; \mathcal{X} \times \mathcal{X} \; \to \; \mathbb{R}$$

of the RKHS H is continuous and bounded

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Main Result: The functional

$$P \ \mapsto \ S\big(P\big) \ = \ \arg\inf_{f \in H} \int L\big(y, f(x)\big)\, P\big(d(x, y)\big) \ + \ \lambda \|f\|_H^2$$

is continuous.

Corollaries

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Corollary 1:

Support vector machines

$$\begin{aligned} S_n \;: \qquad (\mathcal{X} \times \mathcal{Y})^n \quad &\rightarrow \quad H\,, \\ \big((x_1, y_1), \dots, (x_n, y_n)\big) \quad &\mapsto \quad \arg\inf_{f \in H} \frac{1}{n} \sum_{i=1}^{n} L\big(y_i, f(x_i)\big) \;+\; \lambda \|f\|_H^2 \end{aligned}$$

are qualitatively robust.

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Corollary 2:

Support vector machines

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depend on the data **continuously**.

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Corollary 3:

Strong consistency:

$$S_n \xrightarrow{P-\text{a.s.}} S(P) \qquad \text{for} \quad n \to \infty$$

References

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- **R. Hable and A. Christmann (2010)**: Qualitative robustness of support vector machines. *arXiv:0912.0874v1*
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The handout to this talk is also available on my homepage

http://www.staff.uni-bayreuth.de/~btms04/index.html