GOCPS 2008 07.03.2008

Unmeasurable Tests in Statistical Hypothesis Testing

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Hypothesis testing

General testing problem:

$$\mathcal{P}_0 \qquad \text{vs.} \qquad \mathcal{P}_1$$

- Hypotheses $\mathcal{P}_0$ and $\mathcal{P}_1$: Sets of probability measures on $(\Omega, \mathcal{A})$

Simple versus simple:

$$P_0 \qquad \text{vs.} \qquad P_1$$

- Hypotheses P_0 and P_1: Probability measures on $(\Omega, \mathcal{A})$

Task: Find an optimal test!

Optimization problem

Find a level-α-test which minimizes the type II error within the class of all level-α-tests:

$$\int 1 - \varphi \, dP_1 \;=\; \min_{\varphi}! \qquad \text{under} \qquad \int \varphi \, dP_0 \;\leq\; \alpha \tag{1}$$

where the tests $\varphi : \Omega \to [0,1]$ are $\mathcal{A}$- measurable.

$\mathcal{A}$ - **measurability:**

- a rather technical condition
- a rather arbitrary condition

$\mathcal{A}$ - measurability

Practical point of view:

- $\omega \in \Omega$ is the observation; φ' is an *unmeasurable* test
- Deciding according to $\varphi'(\omega)$ is nevertheless possible
- So: $\mathcal{A}$ - measurability is not necessary
- The only problem: How to investigate the performance of the test? $(\mathbb{E}_{P_1}[1-\varphi'],\ \mathbb{E}_{P_0}[\varphi'])$

Theoretical point of view:

- Where do the hypotheses P_0 and P_1 come from?
- P_0 and P_1: no observable objects but mathematical constructs $\longrightarrow$ depend on the chosen sample space
- choise of the sample space rather arbitrary

Topic of the talk

- What happens if unmeasurable tests are permitted?
- How to formulate the optimization problem?
- Is there an unmeasurable test which is better than the best measurable test?
- **Is this an interesting problem?**

Example

$\Omega \ = \ [0,\infty)\,;\quad \omega \in [0,\infty)$ is the outcome of an experiment.
Experimenter not interested in the exact ω but in the interval

$$[k, k+1) \ni \omega\,, \quad k \in \mathbb{N}_0$$

H_0-hypothesis: $P_0\big([k,k+1)\big) = \mathrm{Pois}(\lambda_0)\big(\{k\}\big) \quad \forall\, k \in \mathbb{N}_0$
H_1-hypothesis: $P_1\big([k,k+1)\big) = \mathrm{Pois}(\lambda_1)\big(\{k\}\big) \quad \forall\, k \in \mathbb{N}_0$

- What is the right σ-algebra $\mathcal{A}$?

$$\sigma\Big(\Big\{[k,k+1)\,\Big|\; k \in \mathbb{N}_0\Big\}\Big) \qquad \text{or} \qquad \mathbb{B} \cap [0,\infty)$$

- What are the hypotheses ?

$$\mathrm{Pois}(\lambda_i) \qquad \text{or} \qquad \Big\{P_i \in \mathcal{M}_1(\mathbb{B})\,\Big|\; P_i([k,k{+}1)) = \mathrm{Pois}(\lambda_i)(\{k\})\Big\}$$

Example

A possible setup

- $\mathcal{A} = \sigma\big(\{[k, k+1) \,\big|\, k \in \mathbb{N}_0\}\big)$
- $P_i : \mathcal{A} \to [0,1]\,;\;\; P_i([k, k+1)) = \mathrm{Pois}(\lambda_i)(\{k\}) \quad \forall\, k \in \mathbb{N}_0$

$$P_0 \qquad \text{vs.} \qquad P_1 \qquad\qquad \text{on} \quad (\Omega, \mathcal{A})$$

- permitted tests: $\varphi = \sum_{k \in \mathbb{N}_0} a_k I_{[k,k+1)}$

Another possible setup

- $\mathcal{A}' = \mathbb{B} \cap [0, \infty)$
- $\mathcal{P}_i' = \big\{P_i' : \mathcal{A}' \to [0,1] \,\big|\, P_i'([k, k+1)) = \mathrm{Pois}(\lambda_i)(\{k\}) \;\forall\, k\big\}$

$$\mathcal{P}_0' \qquad \text{vs.} \qquad \mathcal{P}_1' \qquad\qquad \text{on} \quad (\Omega, \mathcal{A}')$$

- permitted φ': Borel-measurable tests

Does the power of the optimal test depend on the setup ?

Mathematical formalization: Outer Expectations

Ω is a set with two σ-algebras

$$\mathcal{A} \subset \mathcal{A}' = 2^{\Omega}$$

H_0-hypothesis: probability measure P_0 on $(\Omega, \mathcal{A})$
H_1-hypothesis: probability measure P_1 on $(\Omega, \mathcal{A})$

Unmeasurable tests not permitted:

$$\mathbb{E}_{P_1}[1-\varphi] = \min_{\varphi}! \qquad \text{under} \qquad \mathbb{E}_{P_0}[\varphi] \le \alpha \tag{2}$$

where the tests $\varphi : \Omega \to [0,1]$ are $\mathcal{A}$- measurable.

Unmeasurable tests permitted:

$$\mathbb{E}^*_{P_1}[1-\varphi'] = \min_{\varphi'}! \qquad \text{under} \qquad \mathbb{E}^*_{P_0}[\varphi'] \le \alpha \tag{3}$$

where the tests $\varphi' : \Omega \to [0,1]$ may be unmeasurable.

Unmeasurable tests not permitted:

$$\beta \ := \ \inf \left\{ \mathbb{E}_{P_1}[1-\varphi] \,\middle|\, \varphi \ \mathcal{A}\text{-meas. such that } \mathbb{E}_{P_0}[\varphi] \leq \alpha \right\}$$

Unmeasurable tests permitted:

$$\beta' \ := \ \inf \left\{ \mathbb{E}^*_{P_1}[1-\varphi'] \,\middle|\, \varphi' \text{ such that } \mathbb{E}^*_{P_0}[\varphi'] \leq \alpha \right\}$$

Obvious: $\beta \ \geq \ \beta'$!

Question: $\beta \ = \ \beta'$?

Results:

- Indeed: $\beta \ = \ \beta'$
- That is: There is no unmeasurable test which is better than the best measurable test
- The same is true in case of general hypothesis testing

$$\mathcal{P}_0 \qquad \text{vs.} \qquad \mathcal{P}_1$$

The proof

$\mathcal{L}_\infty(\Omega)$ set of all bounded functions $f' : \Omega \to \mathbb{R}$
$\mathcal{L}_\infty(\Omega, \mathcal{A})$ set of all bounded, $\mathcal{A}$-measurable functions $f : \Omega \to \mathbb{R}$

A simple idea: Let

$$\kappa : \; \mathcal{L}_\infty(\Omega) \;\to\; \mathcal{L}_\infty(\Omega, \mathcal{A})$$

be a linear, positive, normalized map such that

$$\kappa(f) \;=\; f \qquad \forall\, f \in \mathcal{L}_\infty(\Omega, \mathcal{A}) \subset \mathcal{L}_\infty(\Omega)$$

- Let φ' be an unmeasurable test.

Then:

- $\kappa(\varphi')$ is an $\mathcal{A}$-measurable test
- $\kappa(\varphi')$ is at least as good as φ'

Problem: κ need not exist!

Vector lattices

$\mathcal{L}_\infty(\Omega, \mathcal{A}) \subset \mathcal{L}_\infty(\Omega)$ are M-spaces.

$\kappa : \ \mathcal{L}_\infty(\Omega) \rightarrow \mathcal{L}_\infty(\Omega, \mathcal{A})$ is an M-space projection

κ exists if and only if $\mathcal{L}_\infty(\Omega, \mathcal{A})$ is order complete.

Idea:

- Embed $\mathcal{L}_\infty(\Omega, \mathcal{A})$ into its second dual space

$$\mathcal{L}_\infty(\Omega, \mathcal{A}) \hookrightarrow \big(\mathcal{L}_\infty(\Omega, \mathcal{A})\big)^{**}$$

- The second dual space is an order complete M-space!
- Then, a "generalized projection"

$$\overline{\kappa} : \ \mathcal{L}_\infty(\Omega) \rightarrow \big(\mathcal{L}_\infty(\Omega, \mathcal{A})\big)^{**}$$

does always exist.

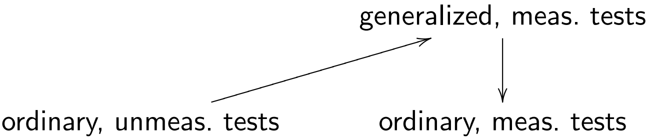
- The "generalized projection"

$$\overline{\kappa} : \ \mathcal{L}_\infty(\Omega) \ \rightarrow \ \big(\mathcal{L}_\infty(\Omega, \mathcal{A})\big)^{**}$$

  leads to a generalized ($\mathcal{A}$-measurable) test

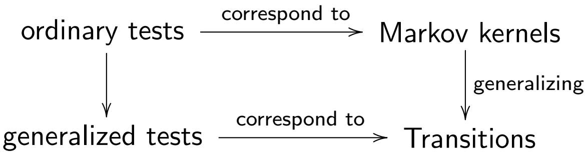
$$\overline{\kappa}(\varphi') \ \in \ \big(\mathcal{L}_\infty(\Omega, \mathcal{A})\big)^{**}$$

- The generalized ($\mathcal{A}$-measurable) test $\overline{\kappa}(\varphi')$ is at least as good as the unmeasurable test φ'
- Main part of the proof: ordinary ($\mathcal{A}$-measurable) tests are as good as generalized ($\mathcal{A}$-measurable) test



Connections to L. Le Cam's general setup

- The use of vector lattices (M-spaces, L-spaces, ...) in statistics is due to Le Cam
- Connections between **generalized tests** and **transitions** (Le Cam (1986)):



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The handout to this talk is also available on my homepage

http://www.statistik.lmu.de/~hable/

Appendix: Generalized tests

An **ordinary** ($\mathcal{A}$-measurable) test $\varphi : \Omega \to [0,1]$ is

$$\varphi \ \in \ \mathcal{L}_\infty(\Omega, \mathcal{A})$$

A **generalized** ($\mathcal{A}$-measurable) test

$$\phi \ \in \ \big(\mathcal{L}_\infty(\Omega, \mathcal{A})\big)^{**}$$

is a map $\phi : \ \mathcal{M}_{1,f}(\mathcal{A}) \ \to \ [0,1]$

$\mathcal{M}_{1,f}(\mathcal{A})$ set of all finitely additive probability measures on $(\Omega, \mathcal{A})$

type I error : $\phi(P_0)$ **type II error** : $1 - \phi(P_1)$

Every ordinary test φ defines a generalized test

$$\phi \ : \ \mathcal{M}_{1,f}(\mathcal{A}) \ \to \ [0,1]\,, \qquad P \ \mapsto \ \mathbb{E}_P[\phi]$$