ICOSSAR 2009 September 14th, 2009

On the Impact of Robust Statistics on Imprecise Probability Models

A Review

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Imprecise Probabilities

Classical probability theory:

- Probabilities specified by exact real numbers $P(A)$
- *"The probability of an event A is*

$$P(A) = 0.4281635907\ldots\text{"}$$

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Imprecise probabilities:

- Probabilities given by lower $L(A)$ and upper bounds $U(A)$
- *"The probability of an event A lies between*

$$L(A) = 0.38 \qquad \textit{and} \qquad U(A) = 0.45\text{''}$$

Imprecise Probabilities: Sets of Probability Measures

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Imprecise probabilities:

- A set of possible probability measures

$$\mathcal{M} \ = \ \Big\{ P : A \mapsto P(A) \ \Big| \ L(A) \leq P(A) \leq U(A) \ \ \forall \, A \Big\}$$

$\mathcal{M}$ is called *structure*.

Imprecise Probabilities and Robust Statistics

Imprecise Probabilities

- Slightly different concepts; developed only recently
  - Walley (1991): coherent lower/upper previsions
  - Weichselberger (2001): interval probabilities
  - ...
- Generalization of classical probabilities
  $\longrightarrow$ model uncertainties about exact, true probabilities
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Close relationship between both areas

Classical Statistics

- A known precise statistical model $\{P_\theta | \theta \in \Theta\}$ is assumed
  - θ: an unknown parameter
  - P_θ: a probability measure depending on the unknown θ
  - True probabilities $P_\theta(A)$ exactly known except for θ
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- strict assumptions lead to unreliable conclusions

Robust Statistics

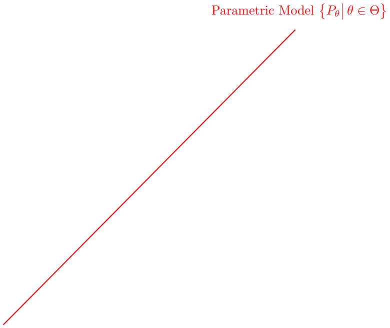
- $\{P_\theta | \theta \in \Theta\}$ is a known precise statistical model
- $\{P_\theta | \theta \in \Theta\}$ is assumed to be approximately true:
  - $U(P_\theta)$: neighborhood about P_θ
  - It is possible that

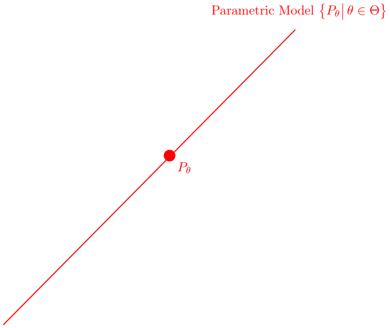
$$\text{the true distribution } P \neq P_\theta$$

  - But: The true distribution P lies in the neighborhood about P_θ

$$P \in U(P_\theta)$$

- Goal: Develop statistical procedures which are still reliable


Parametric Model $\{P_\theta \,|\, \theta \in \Theta\}$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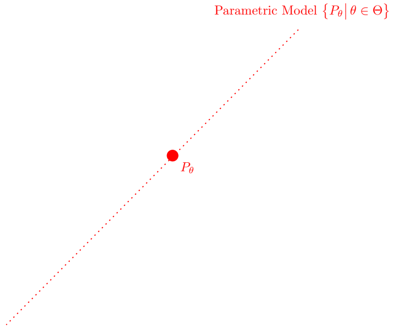
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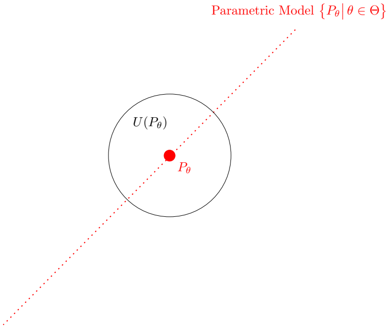
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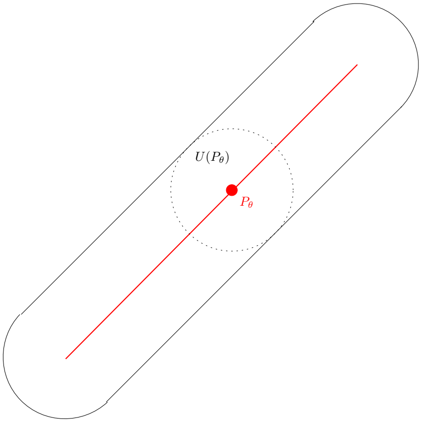
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Robust Statistics and Imprecise Probabilities

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Theorem: *Neighborhoods $U(P_\theta)$ are structures of imprecise probabilities.*

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$\longrightarrow$ Models used in robust statistics are special cases of imprecise probabilities

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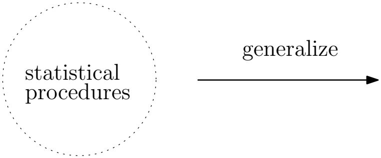
$\longrightarrow$ Models used in robust statistics are special cases of imprecise probabilities

Goal:

Robust Statistics Imprecise Probabilities



Hypothesis Testing

Classical Statistics:

$$H_0 : \quad P = P_0 \qquad \text{vs.} \qquad H_1 : \quad P = P_1$$

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Imprecise Probabilities:

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Hypothesis Testing

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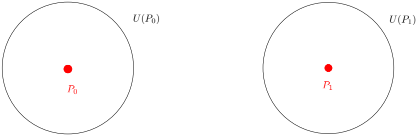
Robust Statistics:



Imprecise Probabilities:

Hypothesis Testing: Least Favorable Pairs

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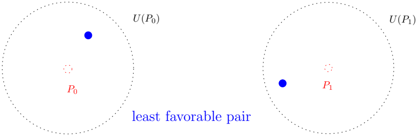
Robust Statistics:



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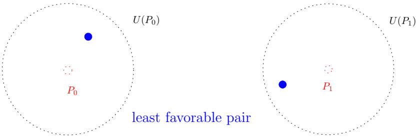

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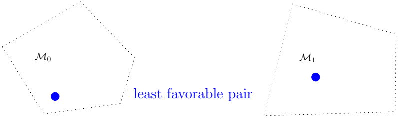

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  - fix the amount of robustness/reliability you want to have
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Imprecise probabilities

- general estimation problems hardly considered explicitly
- “optimal” estimators not available so far

$\longrightarrow$ generalize theory of robust estimating to imprecise probabilities

References

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The handout to this talk is also available on my homepage

http://www.staff.uni-bayreuth.de/~btms04/index.html